



Three positive solutions to a discrete focal boundary value problem

D. Anderson^a, R. Avery^b, A. Peterson^{b,*}

^a *Department of Mathematics and Computer Science, Concordia College Moorhead, MN 56562, USA*

^b *Department of Mathematics and Statistics, University of Nebraska-Lincoln, Lincoln, NE 68588-0323, USA*

Received 15 June 1997

Abstract

We are concerned with the discrete focal boundary value problem $\Delta^3 x(t-k) = f(x(t))$, $x(a) = \Delta x(t_2) = \Delta^2 x(b+1) = 0$. Under various assumptions on f and the integers a , t_2 , and b we prove the existence of three positive solutions of this boundary value problem. To prove our results we use fixed point theorems concerning cones in a Banach space. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Difference equations; Green's function; Fixed points

AMS classification: 39A10

1. Preliminaries

In this section we will state the two fixed point theorems that we will use to prove our main results. First we will make a few definitions.

Definition 1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

- (i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
- (ii) $x \in P, -x \in P$ implies $x = 0$.

The cone $P \subset E$ induces an ordering on the Banach space E given by

$$x \leq y \text{ if and only if } y - x \in P,$$

* Corresponding author. E-mail: apeterso@math.unl.edu

and we say that

$$x < y \text{ whenever } x \leq y \text{ and } x \neq y.$$

Definition 2. An operator is called completely continuous if it is continuous and compact (maps bounded sets into relatively compact sets).

Definition 3. A map α is said to be a nonnegative continuous concave functional on P if

$$\alpha: P \rightarrow [0, \infty)$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

For numbers a, b such that $0 < a < b$, vectors $x, y \in E$ and α a nonnegative continuous concave functional on P we define the following convex sets:

$$P_a = \{x \in P: \|x\| < a\},$$

$$P(\alpha, a, b) = \{x \in P: a \leq \alpha(x), \|x\| \leq b\}$$

and the order interval

$$[x, y] = \{z \in E: x \leq z \leq y\}.$$

The following fixed point theorem is due to Leggett–Williams. The proof of this theorem can be found in [10, 13].

Theorem 4. Let $A: \bar{P}_c \rightarrow \bar{P}_c$ be completely continuous and α be a nonnegative continuous functional on P such that $\alpha(x) \leq \|x\|$ for all $x \in \bar{P}_c$. Suppose there exist $0 < d < a < b \leq c$ such that

- (i) $\{x \in P(\alpha, a, b): \alpha(x) > a\} \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in P(\alpha, a, b)$;
- (ii) $\|Ax\| < d$ for $\|x\| \leq d$;
- (iii) $\alpha(Ax) > a$ for $x \in P(\alpha, a, c)$ with $\|Ax\| > b$.

Then A has at least three fixed points x_1, x_2, x_3 satisfying

$$\|x_1\| < d, \quad a < \alpha(x_2)$$

and

$$\|x_3\| > d \quad \text{and} \quad \alpha(x_3) < a.$$

The proof of the following general fixed point theorem can be found in [4, 10].

Theorem 5. Let X be a bounded closed convex subset of a Banach space E and

$$A: X \rightarrow X$$

be completely continuous. Suppose that X_1, X_2 are disjoint closed convex sets of X and U_1, U_2 are nonempty open sets of X with

$$U_1 \subset X_1 \quad \text{and} \quad U_2 \subset X_2.$$

Moreover, suppose that

$$A(X_1) \subset X_1 \quad \text{and} \quad A(X_2) \subset X_2$$

and that A has no fixed points on

$$X_1 - U_1 \quad \text{and} \quad X_2 - U_2.$$

Then A has at least three distinct fixed points x_1, x_2, x_3 satisfying

$$x_1 \in U_1, \quad x_2 \in U_2$$

and

$$x_3 \in X - (X_1 \cup X_2).$$

2. Introduction to the third-order BVP

We are concerned with proving the existence of three “positive” solutions of the third-order nonlinear focal boundary value problem:

$$-\Delta^3 x(t-k) + f(x(t)) = 0 \quad \text{for all } t \in [a+k, b+k], \quad (1)$$

$$x(a) = \Delta x(t_2) = \Delta^2 x(b+1) = 0, \quad (2)$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous, f is nonnegative for $x \geq 0$ and $k \in \{1, 2\}$.

Avery [6, 7] used techniques similar to those we use here to prove the existence of three positive solutions for different boundary value problems. Agarwal and Wong [1], Eloe and Henderson [9], Henderson [11], and Merdivenci [14, 15] are just a few of the references concerned with the existence of two positive solutions to various boundary value problems.

The solutions of (1), (2) are the fixed points of the operator A defined on the Banach space $(\|\cdot\|$ is the sup norm)

$$E = \{x \mid x: [a, b+3] \rightarrow \mathbb{R}, \quad x(a) = 0\}$$

by

$$Ax(t) = \sum_{s=a+k}^{b+k} G(t,s)f(x(s)),$$

where $G(t, s)$ is the Green's function for the operator L defined by

$$Lx(t) = \Delta^3 x(t - k)$$

with boundary conditions

$$x(a) = \Delta x(t_2) = \Delta^2 x(b + 1) = 0.$$

Define [12] the factorial function by

$$t^{(n)} = t(t - 1) \cdots (t - n + 1)$$

for t a real number and n a positive integer. Then the Green's function is given [5] by

$$G(t, s) = \begin{cases} s \in [a + k, t_2 + k - 1], & \begin{cases} u_1(t, s), & t < s - k + 3, \\ v_1(s), & t \geq s - k + 1, \end{cases} \\ s \in [t_2 + k - 1, b + k], & \begin{cases} u_2(t), & t < s - k + 3, \\ v_2(t, s), & t \geq s - k + 1, \end{cases} \end{cases} \quad (3)$$

where

$$u_1(t, s) = \begin{vmatrix} 0 & t - a & \frac{1}{2}(t - a)^{(2)} \\ -(s - k + 1 - t_2) & 1 & t_2 - a \\ 1 & 0 & 1 \end{vmatrix},$$

$$v_1(s) = \frac{1}{2}(s - k - a + 2)^{(2)},$$

$$u_2(t) = \begin{vmatrix} t - a & \frac{1}{2}(t - a)^{(2)} \\ 1 & t_2 - a \end{vmatrix},$$

$$v_2(t, s) = u_2(t) + \frac{1}{2}(t - s + k - 1)^{(2)}.$$

3. Inequalities and equalities needed in the Existence Theorems

By Anderson [5] if $t_2 - a \geq b - t_2 + 2$,

$$G(t_2, s) \geq G(t, s) > 0,$$

for $t \in (a, b + 3]$, $s \in [a + k, b + k]$. Throughout this paper we assume that

$$t_2 - a > b - t_2 + 2. \quad (4)$$

Lemma 6. For all $t \in [a + 1, b + 3]$ and all $s \in [a + k, b + k]$,

$$\frac{G(t, s)}{G(t_2, s)} \geq \frac{2}{t_2 - a + 1}.$$

Proof. We will consider the quotient in the four cases determined by the branches of the Green's function given in (3).

Case 1: $s \in [a + k, t_2 + k - 1]$, $t \in [a + 1, s - k + 2]$.

For $t \leq s - k + 1$, we have by (3) that

$$\begin{aligned} \Delta_t \frac{G(t,s)}{G(t_2,s)} &= \frac{1}{G(t_2,s)} \Delta_t u_1(t,s) \\ &= \frac{1}{G(t_2,s)} \cdot \begin{vmatrix} 0 & 1 & t-a \\ -(s-k+1-t_2) & 1 & t_2-a \\ 1 & 0 & 1 \end{vmatrix} \\ &= \frac{s-k+1-t}{G(t_2,s)} \\ &\geq 0. \end{aligned}$$

Thus, $G(t,s)/G(t_2,s)$ is nondecreasing for these t , and

$$\frac{G(t,s)}{G(t_2,s)} \geq \frac{G(a+1,s)}{G(t_2,s)}.$$

For $s \leq t_2 + k - 2$,

$$\begin{aligned} \Delta_s \frac{G(a+1,s)}{G(t_2,s)} &= \Delta_s \frac{u_1(a+1,s)}{v_1(s)} \\ &= \frac{v_1(s) \cdot 1 - u_1(a+1,s) \cdot (s-k-a+2)}{v_1(s)v_1(s+1)} \\ &= \frac{\frac{1}{2}(s-k-a+2)^{(2)} + \frac{1}{2}(2a-2s+2k-2)(s-k-a+2)}{v_1(s)v_1(s+1)} \\ &= -\frac{(s-k-a+2)^{(2)}}{2v_1(s)v_1(s+1)} \\ &< 0, \end{aligned}$$

so that $G(a+1,s)/G(t_2,s)$ is decreasing for these s . Hence,

$$\begin{aligned} \frac{G(t,s)}{G(t_2,s)} &\geq \frac{G(a+1,t_2+k-1)}{G(t_2,t_2+k-1)} \\ &= \frac{u_1(a+1,t_2+k-1)}{v_1(t_2+k-1)} \\ &= \frac{t_2-a}{\frac{1}{2}(t_2-a+1)(t_2-a)} \\ &= \frac{2}{t_2-a+1}. \end{aligned}$$

Case 2: $s \in [a+k, t_2+k-1]$, $t \in [s-k+1, b+3]$.

Note that in this case we have

$$\frac{G(t,s)}{G(t_2,s)} = \frac{v_1(s)}{v_1(s)} = 1 > \frac{2}{t_2-a+1}$$

by the second branch of the Green's function in (3).

Case 3: $s \in [t_2 + k - 1, b + k]$, $t \in [a + 1, s - k + 2]$.

For any such s and t ,

$$\begin{aligned}\frac{G(t, s)}{G(t_2, s)} &= \frac{u_2(t)}{u_2(t_2)} \\ &= \frac{(t - a)(2t_2 - a - t + 1)}{(t_2 - a + 1)^{(2)}}.\end{aligned}$$

As a result, $G(t, s)/G(t_2, s)$ is increasing from $t = a + 1$ until $t = t_2$, and then decreasing until $t = s - k + 2$. First, note that

$$\frac{G(a + 1, s)}{G(t_2, s)} = \frac{2}{t_2 - a + 1}.$$

Next, we use the fact that

$$s - k + 2 \geq t_2 + 1$$

for these s to see that

$$\begin{aligned}\frac{G(s - k + 2, s)}{G(t_2, s)} &= \frac{(s - k + 2 - a)(2t_2 - a - s + k - 1)}{(t_2 - a + 1)(t_2 - a)} \\ &\geq \frac{(t_2 + 1 - a)(2t_2 - a - s + k - 1)}{(t_2 - a + 1)(t_2 - a)} \\ &= \frac{2t_2 - a - s + k - 1}{t_2 - a}.\end{aligned}$$

Since in this case $b \geq s - k$, and $2t_2 - a \geq b + 3$ by (4), we get that

$$\frac{2t_2 - a - s + k - 1}{t_2 - a} \geq \frac{2}{t_2 - a}.$$

Consequently,

$$\frac{G(s - k + 2, s)}{G(t_2, s)} \geq \frac{2}{t_2 - a} > \frac{2}{t_2 - a + 1}.$$

Therefore, we again conclude that

$$\frac{G(t, s)}{G(t_2, s)} \geq \frac{2}{t_2 - a + 1}.$$

Case 4: $s \in [t_2 + k - 1, b + k]$, $t \in [s - k + 1, b + 3]$.

We have by (3) that

$$\begin{aligned}\Delta_t \frac{G(t, s)}{G(t_2, s)} &= \frac{1}{G(t_2, s)} \Delta_t v_2(t, s) \\ &= \frac{t_2 - s + k - 1}{G(t_2, s)} \\ &\leq 0.\end{aligned}$$

Hence, $G(t,s)/G(t_2,s)$ is nonincreasing for these t , and its minimum occurs at $t = b + 3$. Then for $s \in [t_2 + k - 1, b + k - 1]$,

$$\begin{aligned} \Delta_s \frac{G(b+3,s)}{G(t_2,s)} &= \frac{1}{u_2(t_2)} \Delta_s v_2(t,s) \\ &= \frac{s - k + 2 - (b+3)}{u_2(t_2)} \\ &= \frac{s - b - k - 1}{u_2(t_2)} \\ &< 0, \end{aligned}$$

so that $G(b+3,s)/G(t_2,s)$ is decreasing for these s . Thus,

$$\begin{aligned} \frac{G(t,s)}{G(t_2,s)} &\geq \frac{G(b+3,b+k)}{G(t_2,b+k)} \\ &= \frac{v_2(b+3,b+k)}{u_2(t_2)} \\ &= \frac{(b+3-a)(2t_2-a-b-2)+2}{(t_2-a)(t_2-a+1)} \\ &\geq \frac{(b+3-a)+2}{(t_2-a)(t_2-a+1)} \\ &> \frac{2}{t_2-a+1}, \end{aligned}$$

since $2t_2 - a - b - 2 \geq 1$ by (4), and $b+3-a > t_2 - a$. \square

Lemma 7. For the Green's function given in (3), for each fixed $t \in [a, b+3]$,

$$\sum_{s=a+k}^{b+k} G(t,s) = \frac{1}{24}(t-a+1)^{(3)} - \frac{1}{8}(t-a)(t+a-2t_2-1)(4b-2t_2-t-a+7).$$

Proof. Fix $t \in [a, t_2 + 1]$. Then, using (3), we obtain

$$\begin{aligned} \sum_{s=a+k}^{b+k} G(t,s) &= \sum_{s=a+k}^{t_2+k-1} G(t,s) + \sum_{s=t_2+k}^{b+k} G(t,s) \\ &= \sum_{s=a+k}^{t+k-2} \frac{1}{2}(s-k-a+2)^{(2)} - \sum_{s=t+k-1}^{t_2+k-1} \frac{1}{2}(t-a)(t+a-2s+2k-3) \\ &\quad - \sum_{s=t_2+k}^{b+k} \frac{1}{2}(t-a)(t+a-2t_2-1) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6}(s-k-a+2)^{(3)} \Big|_{a+k}^{t+k-1} + \frac{1}{2}(t-a) \left(s - \frac{t}{2} - \frac{a}{2} - k + \frac{3}{2} \right)^{(2)} \Big|_{t+k-1}^{t_2+k} \\
&\quad - \frac{1}{2}(t-a)(t+a-2t_2-1)s \Big|_{t_2+k}^{b+k+1} \\
&= \frac{1}{6}(t-a+1)^{(3)} + \frac{1}{2}(t-a) \left(t_2 - \frac{t}{2} - \frac{a}{2} + \frac{3}{2} \right)^{(2)} \\
&\quad - \frac{1}{2}(t-a) \left(\frac{t}{2} - \frac{a}{2} + \frac{1}{2} \right)^{(2)} - \frac{1}{2}(t-a)(t+a-2t_2-1)(b-t_2+1) \\
&= \frac{1}{6}(t-a+1)^{(3)} + \frac{1}{8}(t-a)(t+a-2t_2-3)(t+a-2t_2-1) \\
&\quad - \frac{1}{8}(t-a)(t-a+1)(t-a-1) - \frac{1}{2}(t-a)(t+a-2t_2-1)(b-t_2+1) \\
&= \frac{1}{24}(t-a+1)^{(3)} - \frac{1}{8}(t-a)(t+a-2t_2-1)(4b-2t_2-t-a+7). \tag{5}
\end{aligned}$$

On the other hand, if $t \in [t_2, b+3]$, then

$$\begin{aligned}
\sum_{s=a+k}^{b+k} G(t,s) &= \sum_{s=a+k}^{t_2+k-2} G(t,s) + \sum_{s=t_2+k-1}^{b+k} G(t,s) \\
&= \sum_{s=a+k}^{t_2+k-2} \frac{1}{2}(s-k-a+2)^{(2)} \\
&\quad + \sum_{s=t_2+k-1}^{t+k-3} \left[\frac{1}{2}(s-k-t+2)^{(2)} - \frac{1}{2}(t-a)(t+a-2t_2-1) \right] \\
&\quad - \sum_{s=t+k-2}^{b+k} \frac{1}{2}(t-a)(t+a-2t_2-1) \\
&= \frac{1}{6}(s-k-a+2)^{(3)} \Big|_{a+k}^{t_2+k-1} + \frac{1}{6}(s-k-t+2)^{(3)} \Big|_{t_2+k-1}^{t+k-2} \\
&\quad - \frac{1}{2}(t-a)(t+a-2t_2-1)s \Big|_{t_2+k-1}^{b+k+1} \\
&= \frac{1}{6}(t_2-a+1)^{(3)} - \frac{1}{6}(t_2-t+1)^{(3)} - \frac{1}{2}(t-a)(t+a-2t_2-1)(b-t_2+2). \tag{6}
\end{aligned}$$

Despite appearances, it can be shown that the expressions in (5) and (6) are equal. \square

Define

$$D = \sum_{s=a+k}^{b+k} G(t_2, s)$$

and

$$C = \min_{t \in [a+1, b+3]} \sum_{s=a+k}^{b+k} G(t, s) = \min\{H, K\},$$

where

$$H = \sum_{s=a+k}^{b+k} G(a+1, s)$$

and

$$K = \sum_{s=a+k}^{b+k} G(b+3, s).$$

The constants D and C (and hence H and K) come up in our existence theorems in the next section. One can use Lemma 7 to get that

$$D = \frac{1}{6}(t_2 - a + 1)^{(2)}(3b - 2t_2 - a + 5),$$

$$H = \frac{1}{2}(t_2 - a)(2b - t_2 - a + 3)$$

and

$$K = \frac{1}{24}(b+4-a)^{(3)} - \frac{1}{8}(b+3-a)(b+a-2t_2+2)(3b-2t_2-a+4).$$

4. Theorems on the existence of three positive solutions

In this section we state and prove three theorems concerning the existence of three positive solutions of the BVP (1), (2). By a positive solution of the BVP (1), (2) we mean a solution which is in the cone defined in the proof of the following theorem. In particular, by a positive solution of the BVP (1), (2) we mean a solution $x(t)$ which satisfies $x(a) = 0$ and $x(t) \geq 0$ on $[a, b+3]$.

Theorem 8. Suppose there exist numbers a' and d' where $0 < d' < a'$, such that f satisfies the following properties:

(i) if $x \in [a', \frac{1}{2}(t_2 - a + 1)a']$ then $f(x) > a'/C$,

(ii) if $x \in [0, d']$ then $f(x) < d'/D$ and

(iii) one of the following conditions hold:

(A) $\limsup_{x \rightarrow \infty} f(x)/x < 1/D$;

(B) there exists a number c' such that $c' > \frac{1}{2}(t_2 - a + 1)a'$ and if $x \in [0, c']$ then $f(x) < c'/D$.

Then, the boundary value problem (1), (2) has at least three positive solutions.

Proof. Define

$$\alpha(x) = \min_{t \in [a+1, b+3]} x(t).$$

Note that

$$P = \{x \in E : x(t) \geq 0 \text{ for all } t \in [a, b+3]\}$$

is a cone in the Banach space E with the sup norm, $\alpha : P \rightarrow [0, \infty)$ is a nonnegative continuous concave functional with

$$\alpha(x) \leq \|x\|$$

for all $x \in P$, and the operator A is completely continuous. Also

$$A : P \rightarrow P,$$

since $G(a, s) = 0$ for all $s \in [a+k, b+k]$, $G(t, s) \geq 0$ for all $(t, s) \in [a, b+3] \times [a+k, b+k]$, and f is nonnegative for $x \geq 0$. Let

$$b' = \frac{(t_2 - a + 1)a'}{2}.$$

Claim 1. *If condition (A) holds then there exists a number c' such that $c' > b'$ and $A : \bar{P}_{c'} \rightarrow P_{c'}$.*

Suppose

$$\limsup_{x \rightarrow \infty} \frac{f(x)}{x} < \frac{1}{D},$$

then there exists a $\tau > 0$ and $\sigma < 1/D$ such that if $x > \tau$ then $f(x)/x < \sigma$. Let

$$\beta = \max_{x \in [0, \tau]} f(x),$$

hence we have

$$f(x) \leq \sigma x + \beta$$

for all $x \geq 0$. Let

$$c' > \max \left\{ \frac{\beta}{1/D - \sigma}, b' \right\}.$$

Thus, if $x \in \bar{P}_{c'}$ we have,

$$\begin{aligned} \|Ax\| &= \max_{t \in [a, tb+3]} \sum_{s=a+k}^{b+k} G(t, s) f(x(s)) \\ &= \sum_{s=a+k}^{b+k} G(t_2, s) f(x(s)) \\ &\leq \sum_{s=a+k}^{b+k} G(t_2, s) (\sigma \|x\| + \beta) \\ &\leq \sum_{s=a+k}^{b+k} G(t_2, s) (\sigma c' + \beta) \\ &= (\sigma c' + \beta) D < c'. \end{aligned}$$

Thus, $A : \bar{P}_{c'} \rightarrow P_{c'}$ and Claim 1 has been shown.

Claim 2. *If there exists a positive number r such that $x \in [0, r]$ implies*

$$f(x) < \frac{r}{D},$$

then

$$A : \bar{P}_r \rightarrow P_r.$$

Suppose that $x \in \bar{P}_r$, thus

$$\begin{aligned} \|Ax\| &= \sum_{s=a+k}^{b+k} G(t_2, s) f(x(s)) \\ &< \sum_{s=a+k}^{b+k} G(t_2, s) \frac{r}{D} = r. \end{aligned}$$

Therefore, $A : \bar{P}_r \rightarrow P_r$ and Claim 2 has been shown.

Hence, we have shown in the previous claims that if either condition (A) or condition (B) holds, then there exists a number c' such that $c' > b'$ and $A : \bar{P}_{c'} \rightarrow P_{c'}$. Note from Claim 2 with $r = d'$ we get using (ii) that $A : \bar{P}_{d'} \rightarrow P_{d'}$.

Claim 3. $\{x \in P(\alpha, a', b') : \alpha(x) > a'\} \neq \emptyset$ and $\alpha(Ax) > a'$ for all $x \in P(\alpha, a', b')$.

We have

$$x(t) := \frac{b' + a'}{2} \in \{x \in P(\alpha, a', b') : \alpha(x) > a'\},$$

hence it is nonempty. Let $x \in P(\alpha, a', b')$; then using (i),

$$\begin{aligned}\alpha(Ax) &= \min_{t \in [a+1, b+3]} \sum_{s=a+k}^{b+k} G(t, s) f(x(s)) \\ &> \min_{t \in [a+1, b+3]} \sum_{s=a+k}^{b+k} G(t, s) \frac{a'}{C} \\ &= \left(\frac{a'}{C}\right) \min_{t \in [a+1, b+3]} \sum_{s=a+k}^{b+k} G(t, s) = a'.\end{aligned}$$

Thus, if $x \in P(\alpha, a', b')$ then $\alpha(Ax) > a'$.

Claim 4. If $x \in P(\alpha, a', c')$ and $\|Ax\| > b'$ then $\alpha(Ax) > a'$.

Suppose $x \in P(\alpha, a', c')$ and $\|Ax\| > b'$, then

$$\begin{aligned}\alpha(Ax) &= \min_{t \in [a+1, b+3]} \sum_{s=a+k}^{b+k} G(t, s) f(x(s)) \\ &= \min_{t \in [a+1, b+3]} \sum_{s=a+k}^{b+k} \frac{G(t, s)}{G(t_2, s)} G(t_2, s) f(x(s)) \\ &\geq \sum_{s=a+k}^{b+k} \left(\frac{2}{t_2 - a + 1}\right) G(t_2, s) f(x(s)) \\ &= \left(\frac{2}{t_2 - a + 1}\right) \sum_{s=a+k}^{b+k} G(t_2, s) f(x(s)) \\ &= \left(\frac{2}{t_2 - a + 1}\right) \|Ax\| > a'.\end{aligned}$$

Thus, $\alpha(Ax) > a'$.

Hence, the hypotheses of the Leggett–Williams existence theorem are satisfied; therefore (1), (2) has at least three positive solutions. \square

From the proof of the above theorem and the last statement in Theorem 4 we get that there are positive solutions x_1 , x_2 and x_3 for the focal boundary value problem (1), (2) (guaranteed by Theorem 8) such that

$$\|x_1\| < d',$$

$$a' < x_2(t) < c' \text{ for } t \in [a+1, b+3],$$

and

$$\max_{t \in [a, b+3]} x_3(t) > d' \text{ with } \min_{t \in [a+1, b+3]} x_3(t) < a'.$$

Theorem 9. Suppose there exist numbers d', a' and b' where $0 < d' < a' < b'$, such that f satisfies the following properties:

- (i) if $x \in [a', b']$ then $f(x) > \frac{a'}{(t_2 - a)(t_2 - a + 2 - k)}$,
 - (ii) if $x \in [0, d']$ then $f(x) < d'/D$ and
 - (iii) one of the following conditions hold:
 - (A) $\limsup_{x \rightarrow \infty} f(x)/x < 1/D$;
 - (B) there exists a number c' such that $c' > b'$ and if $x \in [0, c']$ then $f(x) < c'/D$.
- Then, the boundary value problem (1), (2) has at least three positive solutions.

Proof. Define

$$\alpha(x) := \min_{t \in [t_2, t_2+1]} x(t).$$

We need to verify conditions (i) and (iii) in the Leggett–Williams Theorem, the remaining conditions were verified in the proof of Theorem 8. Thus, from the proof of Theorem 8, there exists $c' > b'$ such that

$$A : \bar{P}_{c'} \rightarrow P_{c'}.$$

Claim 1. $\{x \in P(\alpha, a', b') : \alpha(x) > a'\} \neq \emptyset$ and $\alpha(Ax) > a'$ for all $x \in P(\alpha, a', b')$.

We have

$$x(t) := \frac{b' + a'}{2} \in \{x \in P(\alpha, a', b') : \alpha(x) > a'\},$$

hence it is nonempty. Let $x \in P(\alpha, a', b')$, then using $\Delta_t G(t_2, s) = 0$,

$$\begin{aligned} \alpha(Ax) &= \min_{t \in [t_2, t_2+1]} \sum_{s=a+k}^{b+k} G(t, s) f(x(s)) \\ &= \sum_{s=a+k}^{b+k} G(t_2, s) f(x(s)) \\ &\geq \sum_{s=t_2}^{t_2+1} G(t_2, s) f(x(s)) \\ &> \left(\frac{a'}{(t_2 - a)(t_2 - a + 2 - k)} \right) \sum_{s=t_2}^{t_2+1} G(t_2, s) \\ &= \left(\frac{a'}{(t_2 - a)(t_2 - a + 2 - k)} \right) (t_2 - a)(t_2 - a + 2 - k) = a'. \end{aligned}$$

Thus, if $x \in P(\alpha, a', b')$ then $\alpha(Ax) > a'$.

Claim 2. If $x \in P(\alpha, a', c')$ and $\|Ax\| > b'$ then $\alpha(Ax) > a'$.

Suppose $x \in P(\alpha, a', c')$ and $\|Ax\| > b'$, then

$$\begin{aligned}\alpha(Ax) &= \min_{t \in [t_2, t_2+1]} \sum_{s=a+k}^{b+k} G(t, s) f(x(s)) \\ &= \sum_{s=a+k}^{b+k} G(t_2, s) f(x(s)) \\ &= \|Ax\| > b' > a'.\end{aligned}$$

Thus, $\alpha(Ax) > a'$.

Hence, the hypotheses of the Leggett–Williams existence theorem are satisfied, therefore (1), (2) has at least three positive solutions. \square

From the proof of the above theorem and the last statement in Theorem 4 we get that there are positive solutions x_1 , x_2 , and x_3 for the focal boundary value problem (1), (2) (guaranteed by Theorem 8) such that

$$\|x_1\| < d',$$

$$a' < x_2(t_2), \quad x_2(t_2 + 1) < c'$$

and

$$\|x_3\| > d' \text{ with either } x_3(t_2) < a' \text{ or } x_3(t_2 + 1) < a'.$$

Theorem 10. Suppose there exist numbers a', b' and d' where $0 < d' < a' < b'$, such that f satisfies the following properties:

- (i) if $x \in [a', b']$ then $a'/C < f(x) < b'/D$,
- (ii) if $x \in [0, d']$ then $f(x) < d'/D$ and
- (iii) one of the following conditions hold:

$$(A) \limsup_{x \rightarrow \infty} f(x)/x < 1/D;$$

$$(B) \text{ there exists a number } c' \text{ such that } c' > b' \text{ and if } x \in [0, c'] \text{ then } f(x) < c'/D.$$

Then, the boundary value problem (1), (2) has at least three positive solutions.

Proof. As before P is a cone in the Banach space E with the sup norm, A is completely continuous, and $A : P \rightarrow P$. We have shown in Theorem 8 that condition (iii) implies there exists a

$$c' > b',$$

such that

$$A : \bar{P}_{c'} \rightarrow P_{c'},$$

and condition (ii) implies that

$$A : \bar{P}_{d'} \rightarrow P_{d'}.$$

For a real number w define

$$x_w(t) := \begin{cases} w, & t \in [a+1, b+3], \\ 0, & t = a. \end{cases}$$

Define the order interval $X_1 = [x_{a'}, x_{b'}]$ and let U_1 be the interior of X_1 in \bar{P}_c . Note that

$$x_{((a'+b')/2)} \in U_1,$$

hence U_1 is nonempty.

Claim. $A(X_1) \subset X_1$ and A has no fixed points in $X_1 - U_1$.

Suppose $x_{a'} \leq x \leq x_{b'}$ and $t \in [a+1, b+3]$, then

$$\begin{aligned} Ax(t) &= \sum_{s=a+k}^{b+k} G(t,s)f(x(s)) \\ &> \sum_{s=a+k}^{b+k} G(t,s) \left(\frac{a'}{C} \right) \\ &= \left(\frac{a'}{C} \right) \sum_{s=a+k}^{b+k} G(t,s) \\ &\geq \left(\frac{a'}{C} \right) \min_{t \in [a+1, b+3]} \sum_{s=a+k}^{b+k} G(t,s) = a' \end{aligned}$$

Also,

$$\begin{aligned} Ax(t) &= \sum_{s=a+k}^{b+k} G(t,s)f(x(s)) \\ &< \sum_{s=a+k}^{b+k} G(t,s) \left(\frac{b'}{D} \right) \\ &= \left(\frac{b'}{D} \right) \sum_{s=a+k}^{b+k} G(t,s) \\ &\leq \left(\frac{b'}{D} \right) \sum_{s=a+k}^{b+k} G(t_2,s) = b'. \end{aligned}$$

Hence, for all $t \in [a+1, b+3]$,

$$x_{a'}(t) = a' < Ax(t) < b' = x_{b'}(t),$$

thus it follows that $Ax \in U_1$. This also implies that A has no fixed points on $X_1 - U_1$ since $A(X_1) \subset U_1$.

Thus, the hypotheses of Theorem 5 are satisfied with

$$X = \bar{P}_{c'}, \quad X_2 = \bar{P}_{d'} \quad \text{and} \quad U_2 = P_{d'}. \quad \square$$

From the proof of the above theorem and the last statement in Theorem 5 we get that there are positive solutions x_1 , x_2 and x_3 for the focal boundary value problem (1), (2) (guaranteed by Theorem 10) such that:

$$\|x_1\| < d',$$

$$a' < x_2(t) < b' \quad \text{for } t \in [a+1, b+3],$$

and

$$\|x_3\| > d' \quad \text{with } x_{a'} \not\leq x_3 \not\leq x_{b'}.$$

References

- [1] R.P. Agarwal, P.J.Y. Wong, Double positive solutions of (n, p) boundary value problems for higher-order difference equations, *Comput. Math. Appl.* 32 (1996) 1–21.
- [2] C.D. Ahlbrandt, A.C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions and Riccati Equations*, Kluwer Academic Publishers, Boston, 1996.
- [3] H. Amann, On the number of solutions of nonlinear equations in ordered Banach spaces, *J. Funct. Anal.* 11 (1972) 346–384.
- [4] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* 18 (1976) 620–709.
- [5] D. Anderson, Green's function for an n -point discrete right focal boundary value problem, *PanAmer. Math. J.* 8 (1998).
- [6] R. Avery, Multiple positive solutions of an n th order focal boundary value problem, *PanAmer. Math. J.* 8 (1998).
- [7] R. Avery, Three positive solutions of a discrete second order conjugate problem, *PanAmer. Math. J.*, submitted.
- [8] K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, 1985.
- [9] P. Eloe, E. Kaufmann, J. Henderson, Multiple positive solutions for difference equations, *J. Difference Equations Appl.*
- [10] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.
- [11] J. Henderson, Positive solutions for nonlinear difference equations, *Nonlinear Studies* 4 (1997) 29–36.
- [12] W.G. Kelley, A.C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, San Diego, 1991.
- [13] R.W. Leggett, L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana University Math. J.* 28 (1979) 673–688.
- [14] F. Merdivenci, Two positive solutions of a boundary value problem for difference equations, *J. Difference Equations Appl.* 1 (1995) 263–270.
- [15] F. Merdivenci, Green's matrices and positive solutions of a discrete boundary value problem, *PanAmer. Math. J.* 5 (1995) 25–42.